A UNIFIED TREATMENT OF DERIVATIVE PRICING AND FORWARD DECISION PROBLEMS WITHIN HJM FRAMEWORK

by

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ABSTRACT

WENHUA ZOU. A unified treatment of derivative pricing and forward decision problems within HJM framework. (Under direction of DR. MINGXIN XU.)

We study the HJM approach which was originally introduced in the fixed income market by David Heath, Robert Jarrow and Andrew Morton and later was implemented in the case of European option market by Martin Schweizer, Johannes Wissel, Rene Carmona and Sergey Nadtochiy. The main contribution of this thesis is to apply HJM philosophy to the American option market. We derive the absence of arbitrage by a drift condition and compatibility between long and short rate by a spot consistency condition. In addition, we introduce a forward stopping rule which is significantly different from the classical stopping rule which requires backward induction. When Itô stochastic differential equation are used to model the dynamics of underlying asset, we discover that the drift part instead of the volatility part will determine the value function and stopping rule. As counterpart to the forward rate for the fixed income market and implied forward volatility and local volatility for the European option market, we introduce the forward drift for the American option market.

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CHAPTER 1: HJM PHILOSOPHY

Modeling is a very important issue for the derivative market. Given a model, we can do the pricing and hedging. Because the initial values of the bond and option price for different maturities are observable from the market, the first requirement for a model is to be consistent with the initial observations. Since many spot rate models have some strong assumptions for their coefficients, for example Vasicek Model for interest rate market and Black Scholes Model for option market, values generated by these models can not match the initial observations. Even those models which let coefficients depend on time requires frequent recalibration. In addition, there is no theoretical solution to when to do the recalibration. Heath, Jarrow and Morton proposed to solve the problem by modelling directly the dynamics of the entire structure of the interest rate curve. Because the initial prices of European option for different maturities are also observable from the market, HJM philosophy was extended to model the dynamics of forward implied volatility by M. Schwerizer and J. Wissel (2008), R. Carmona, S. Nadtochiy (2009, 2011).

In this chapter, we will summarize how HJM philosophy is applied to the fixed income and the European option market. In section one, we will introduce the forward rate model for fixed income market. In section two, we will summarize the implied volatility model and local volatility model for European option market. The goal of this chapter is to introduce the main concepts for HJM model such as spot condition, drift condition and how they are related to the spot rate models.

1.1 Fixed Income Market

Given filtered probability space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P)$, where $(\mathcal{F}_{t\geq 0})$ satisfies the usual condition and P is the risk-neutral probability measure.

Definition 1.1 *Given continuous-time Markov process (short rate)* $\{r_t\}_{t\geq 0}$, *define:*

1. $P(t,T) = \mathbb{E}_t e^{-\int_t^T r_s ds}$.

   $P(t,T)$ is the price of zero coupon bond at t with maturity T.

   Note: $P(0,T)$ can be observed for different maturities T.

2. $B_t = e^{\int_0^t r_s ds}$.

   B_t is the bank account.

Definition 1.2 Suppose $P(t,T)$ is smooth in the maturity variable T, then define the forward rate as

$$f_t(T) = \frac{\partial}{\partial T} \log P(t,T).$$

Lemma 1.3 [Spot Consistency Condition]

$$\text{For all } t \geq 0, f_t(t) = r_t.$$

Proof. Taking derivative with respect to T, we can get

$$\frac{\partial P(t,T)}{\partial T} = \mathbb{E}_t \frac{\partial e^{-\int_t^T r_s ds}}{\partial T} = \mathbb{E}_t[-r_T \cdot e^{-\int_t^T r_s ds}].$$

On the other hand

$$P(t,T) = e^{-\int_t^T f_t(u) du}$$

which is equivalent to

$$f_t(T) = -\frac{\partial}{\partial T}\log P(t,T)$$

$$\frac{\partial e^{-\int_t^T f_t(u)du}}{\partial T} = -f_t(T)e^{-\int_t^T f_t(u)du}$$

Set $T = t$, we can get: for $t \geq 0$,

$$f_t(t) = r_t$$

◇

1.1.1 Forward Rate Model

Recall the relationship between B_t and r_t:

(1) $$dB_t = r_t B_t dt$$

with initial value $B_0 = 1$.

Forward rate model is formed as following:

$$dB_t = \begin{cases} r_t B_t dt, & B_0 = 1 \\ f_t(t) = r_t, & for\ t \geq 0 \\ df_t(u) = \alpha_t(u)dt + \beta_t(u)dW_t, & f_0(u) \end{cases} \quad (2)$$

where $\alpha_t(u)$ and $\beta_t(u)$ satisfy that $f_t(u)$ has a unique strong solution. The above model should be built to satisfy:

1. Initial observation of the bond price $P(0,T)$ for all $T \geq 0$ from the market can be reproduced by the model.

2. The model should be arbitrage free.

The first requirement can be included in the initial value of forward rate $f_0(T)$ such that $f_0(T) = -\frac{\partial}{\partial T}\log P(0,T)$. The second requirement will give us the famous HJM "drift condition". We explain below that it is the consequence of enforcing the martingale property. Since $P(t,T)$ is a martingale under the risk neutral measure P, this martingale property leads to a constraint which is known under name of "drift condition".

Theorem 1.4 Recall the definition of $\beta_t(u)$ and $\alpha_t(u)$.

For all $0 \leq t \leq T$.

$$\alpha_t(T) = \beta_t(T) \cdot \int_t^T \beta_t(s)ds$$

Proof of this theorem can be found in Heath, Jarrow and Morton [4].

The above formula shows that drift is completely determined by volatility.

The procedure to apply this HJM is: First we model the volatility of the forward rate. Second, we calculate the drift of this forward rate.

Example 1.5:

Suppose $\beta_t(T) = \sigma f_t(T)$, then according to the theorem above, we can have $\alpha_t(T) = \beta_t(T) \cdot \int_t^T \beta_t(u)du = \sigma^2 f_t(T) \int_t^T f_t(u)du$. Heath, Jarrow and Morton [5] shows that this drift condition causes forward rates to explode.

Example 1.6 Shreve [28] gives the following example:

Suppose $\beta_t(T) = s(t)\sigma(T-t)\min\{M, f_t(T)\}$, where $s(t)$, $\sigma(T-t)$ are deterministic function and M is a constant number.

Then they got:

$$\begin{aligned}\alpha_t(T) &= \beta_t(T) \cdot \int_t^T \beta_t(u)du \\ &= s(t)^2\sigma(T-t)\min\{M, f_t(T)\}\int_t^T \sigma(u-t)\min\{M, f_u(T)\}\,du\end{aligned}$$

Given forward rate model $f_t(u)$, we can get $r_t = f_t(t)$. On the other hand, given spot rate model r_t, calculating the $f_t(T)$ is not easy. We will need to calculate $P(0,T) = \mathbb{E}e^{-\int_0^T r_s ds}$ first and then $f_0(T) = -\frac{\partial}{\partial T} log P(0,T)$. In most cases, we can have the analytic solution of $f_0(T)$ only if analytic solution of $P(0,T)$ is available. The following is an example of Affine models.

Example 1.7 Vasicek model:

$$dr_t = (\alpha - \beta r_t)dt + \sigma dW_t$$

where α and σ are constants. Then

$$P(0,T) = e^{A(T)+B(T)r_0} \tag{3}$$

where

$$A(T) = \frac{4\alpha\beta - 3\sigma^2}{4\beta^2} + \frac{\sigma^2 - 2\alpha\beta}{2\beta^2}T + \frac{\sigma^2 - \alpha\beta}{\beta^3}e^{-\beta T} - \frac{\sigma^2}{4\beta^3}e^{-2\beta T}$$

and

$$B(T) = -\frac{1}{\beta}(1 - e^{-\beta T})$$

In addition, we can get:

$$f_t(T) = r_t e^{-\beta(T-t)} + \frac{\alpha}{\beta}(1 - e^{-\beta(T-t)}) - (1 - e^{-\beta(T-t)})\frac{\sigma^2}{2\beta^2} \tag{4}$$

In practice, factor models are very popular. we discuss factor models using Nelson and Siegel model as an example. We quote the description from the summary in R. Carmona [2005].

Just like HJM approach, martingale property is used to give the no-arbitrage condition. "*A Factor model starts from a function* G *from* $\Theta \times [0, \infty)$ *into* $[0, \infty)$ *where* Θ *is an open set in* R^d *which we interpret as the set of possible values of a vector of parameters* $\theta^1, \theta^2, ...\theta^d$*. Then* $G(\theta, .) : \tau \to G(\theta, \tau)$ *can be viewed as a possible candidate for the forward curve. Nelson and Siegel has three parameters as*

$$G(\theta, \tau) = \theta^1 + (\theta^2 + \theta^3\tau)e^{-\theta^4\tau}, \tau \geq 0$$

and

$$d\theta^i_t = b^i_t \cdot dt + \sum_{j=1}^{D} \sigma \cdot dW^j_t$$

with initial value θ^i_0*.*

Here θ^i_0 *is* $\mathcal{F}_0-$*measurable, and* b *and* σ *are progressively measurable process with values in* R^4 *and* R^{4*D} *respectively, such that* $\int_0^t (|b_s| + |\sigma_s|)^2 ds < \infty$*,* $P-$*almost surely for all finite* t*. Assuming further that* G *is twice continuously differentiable in the variables* θ^j *, we can use Ito's formula and derive the dynamics of* $\overline{f}_t(\tau)$*. The parameters* θ_1*and* θ_4 *are assumed to be positive.* θ_1 *represents the asymptotic (long) forward rate,* $\theta_1 + \theta_2$ *gives the left end point of the curve, namely the short rate, while* θ_4 *gives an asymptotic rate of decay. The set* Θ *of parameters is the subset of* R^4 *determined by* $\theta_1 > 0, \theta_4 > 0$ *and* $\theta_1 + \theta_2 > 0$ *since the short rate should not be negative. The parameter* θ_3 *is responsible for a hump when* $\theta_3 > 0$ *or a dip with* $\theta_3 < 0$*.*"

1.2 European Option Market

As is well known, the Black Scholes model is used to model the underlying asset to price the European option. However, volatility is assumed to be a constant number in the model which is totally different from the observation from the market. In fact, implied volatility is a function of both time to maturity and strike price. Many approaches have been created to solve this problem, for example implied volatility model and local volatility model. In this part, we summarize two recent developments that apply the HJM philosophy to those models: implied forward volatility model by Schweizer, Wissel (2008) and local volatility dynamic model by Carmona, Nadtochiy (2009).

1.2.1 Implied Forward Volatility

Given Probability space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P)$ where $(\mathcal{F}_{t\geq 0})$ satisfies the usual condition and P is the risk-neutral measure. The spot volatility model is

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \tag{5}$$

where $\{\mu_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are adapted stochastic processes to be specified. In addition, $W = \{W_t\}_{t\geq 0}$ is a $d-$dimensional Wiener process.

Schweizer, Wissel (2008) introduced forward implied volatilities $X(t,T)$ defined by

$$X(t,T) = \frac{\partial}{\partial T}((T-t)\Sigma_t(T)^2), \tag{6}$$

where $\Sigma_t(T)$ is the implied volatility.

The implied forward volatility model is:

$$dS_t = \begin{cases} \mu_t S_t dt + \sigma_t S_t dW_t, & S_0 \\ X_t(t) = \sigma_t, & for \quad t \geq 0 \\ dX_t(u) = \alpha_t(u)dt + \beta_t(u)dW_t, & X_0(u) \end{cases} \quad (7)$$

Then they proved the Spot Consistency Condition $X_t(t) = \sigma_t$ for $t \geq 0$ and Drift Condition in proposition 2.2 and theorem 2.1 in their paper.

1.2.2 Local Volatility Model

The spot volatility model is

$$dS_t = \sigma_t S_t dW_t \quad (8)$$

where $\{\mu_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are adapted stochastic processes to be specified. In addition, $W = \{W_t\}_{t\geq 0}$ is $d-$dimensional Wiener process.

Carmona, Nadtochiy (2009) used local volatility which was introduced by Dupire (1994):

$$a_t^2(\tau, K) = \frac{2\partial C_t(\tau, K)}{K^2 \partial_{KK}^2 C_t(\tau, K)},$$

for $\tau > 0$ and $K > 0$. Here $C_t(\tau, K)$ is the value of call option at time t with the maturity $t + \tau$.

The local volatility model is:

$$dS_t = \begin{cases} \sigma_t S_t dW_t, & S_0 \\ a_t(0) = \sigma_t, & for \quad t \geq 0 \\ da_t^2(\tau, x) = a_t^2(u)[\alpha_t(\tau, x)dt + \beta_t(\tau, x)dW_t], & a_0^2(\tau, x) \end{cases} \quad (9)$$

Carmona, Nadtochiy (2009) gave the drift condition in theorem 4.1 in their paper. The following are two examples among those given in the paper that demonstrate the computation between the short rate and the forward rate model.

Example 1.8 Suppose $\beta_t(\tau, x) = 0$ for all $\tau > 0$ and $x > 0$.

According to the drift condition, we can get: $a_t(\tau, x) = a_0(\tau + t, x)$. Therefore

$$\sigma_t = a_0(t, logS_t).$$

Example 1.9

$$dS_t = \begin{cases} S_t r dt + S_t \sigma_t(\sqrt{1-\rho^2} dB_t^1 + \rho dB_t^2), & S_0 \\ d\sigma_t = f(t, \sigma_t)dt + g(t, \sigma_t)dB_t^2, & \sigma_0 \end{cases}$$

where B_t^1 and B_t^2 are independent Brownian motions, $\rho \in [-1, 1]$. $f(t, x)$ and $g(t, x)$ satisfy the usual conditions which guarantee the existence and uniqueness of a positive solution to the above system.

Carmona, Nadtochiy (2009) proved that the local volatility surface is given at time $t = 0$ by the formula

$$a^2(T, K) = \frac{[\sigma_T^2 \frac{\overline{S}_T}{\overline{\sigma}_T} e^{-\frac{d_1^2(T,K)}{2}}]}{E[\frac{\overline{S}_T}{\overline{\sigma}_T} e^{-\frac{d_1^2(T,K)}{2}}]}.$$

As in the fixed income market, factor models are very popular in practice for the equity market. Brigo and Mercurio in [7, 8] introduced the following factor model. You can also find following summary in Carmona [2008]:

$$\text{“}\Theta = (\sigma, \eta_1, \eta_2, \theta_1, \theta_2, p_1, p_2, s, u)$$

satisfying condition: $p_1,\ p_2 > 0, p_1 + p_2 \leq 1,\ \theta_1, \theta_2 \geq 0,\ \sigma > 0,\ \mu \geq 0$ *Let*

$$v_i(\tau) = \sqrt{\theta_i + (\sigma^2 - \theta_i)\frac{1 - e^{-\eta_i \tau}}{\eta_i \tau}}$$

and

$$d_i(\tau, x) = \frac{s - x + (\mu + \frac{1}{2}v_i^2(\tau))\tau}{\sqrt{\tau} v_i(\tau)}$$

and $\eta_0 = 0, p_0 = 1 - p_1 - p_2,\ v_0(\tau) = \sigma, d_0(\tau, x) = \frac{s - x + (\mu + \frac{1}{2}\sigma^2)\tau}{\sqrt{\tau}\sigma}$ *Then*

$$a^2(\Theta, \tau, x) = \frac{\sum_{i=0}^2 p_i(\theta_i + (\sigma^2 - \theta_i)e^{-\eta_i \tau})exp(-\frac{d_i^2(\tau,x)}{2})/v_i(\tau)}{\sum_{i=0}^2 p_i exp(-\frac{d_i^2(\tau,x)}{2})/v_i(\tau)}$$

The meaning of each of the parameters is as follows:

s *is the logarithm of the current stock price.*

σ *is the spot volatility.*

μ *is the drift of the stock process (most likely, the difference between interest rate and the dividend payment rate).*

$\{\eta_i, \theta_i\}_1^2$ *define scenarios for the volatility process.*

p_i *are the respective probabilities of these scenarios."*

CHAPTER 2: OPTIMAL STOPPING PROBLEM

Classic optimal stopping problem is well-studied with nice results using martingale or Markovian approach. We refer to Oksendal (2004), Peskir, Shiryaev(2006), Villenenve (2007) and Dayanik, Karatzas (2008) for classical accounts of the theory. For the classical problem, the philosophy of backward induction is used to solve the optimal stopping time. For discrete-time case, Wald-Bellman equation is used to find the optimal solution. For continuous-time case, Wald-Bellman equation changes to Snell Envelope.

There is vast number of literature on the application of optimal stopping problem: for example optimal stock selling time by Zhang(2001), Guo and Liu (2005); option pricing problem by Guo and Shepp (2001), Carmona and Touzi (2008); search problems by Nishimura and Ozaki (2004); optimal stopping problem with multiple priors by Riedel (2009).

The contribution for this chapter is to extend two verification theorems to two optimal stopping times problems. In addition, they are extended to two optimal stopping times problems. We begin with the introduction of the discrete and continuous optimal stopping time problem in section one. In section two, we will prove two verification theorems for optimal stopping time problems. In section three, we will extend the verification theorems from classic one optimal to two optimal stopping time problems. In section four, examples are given based on the drifted Brownian motion and geometric Brownian motion.

2.1 Discrete And Continuous Results

Discrete Case:

Let $G = (G_n)_{n\geq 0}$ be a sequence of random variables defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$. G is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$, in the sense that each G_n is $\mathcal{F}_n$-measurable. Recall that each $\mathcal{F}_n$ is σ-algebra if subsets of Ω such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}$. Typically $(\mathcal{F}_n)_{n\geq 0}$ coincides with the natural filtration $(\mathcal{F}_n^G)_{n\geq 0}$

Definition 2.1 A random variable $\tau : \Omega \to \{0, 1, ..., \infty\}$ is called Markov time if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $0 \leq n \leq N$. A Markov time is called a stopping time if $\tau < \infty$ $P.a.s.$ The family of all stopping times will be denoted by M.

Definition 2.2 $M_n^N = \{\tau \in M : n \leq \tau \leq N\}$

Assumption 2.3

$$\mathbb{E}(\sup_{0\leq k\leq N} |G_k|) < \infty$$

for all $N > 0$ with $G_N \equiv 0$ when $N = \infty$.

Consider the optimal stopping time:

$$V_N = \sup_{0\leq \tau\leq N} \mathbb{E}G_\tau \tag{10}$$

where $0 \leq N$ and τ is a stopping time.

Definition 2.4

$$S_n^N = \begin{cases} G_N, & for \qquad n = N \\ \max\left[G_n, \mathbb{E}\left[S_{n+1}^N \,|\mathcal{F}_n\right]\right], & for \quad n = N-1, ..., 0. \end{cases}$$

Definition 2.5

$$\tau_n^N = \inf\left\{n \leq k \leq N : S_k^N = G_k\right\}$$

for $0 \leq n \leq N$. Note that the infimum above is always attainedimum above is always attained

Theorem 2.6 Finite horizon

Consider the optimal stopping problem [10] with $N < \infty$ upon assuming that [2.3] holds. Then for $0 \leq n \leq N$ we have:

$$S_n^N \geq \mathbb{E}(G_\tau|\mathcal{F}_n) \tag{11}$$

for each $\tau \in m_n^N$.

$$S_n^N \geq \mathbb{E}(G_{\tau_n^N}|\mathcal{F}_n) \tag{12}$$

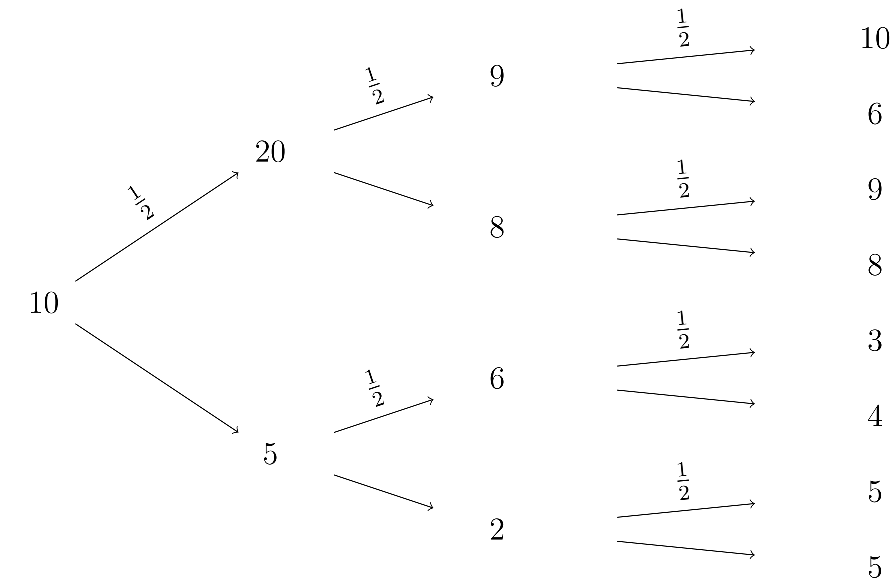
Moreover, we can have

1. The stopping time τ_0^N is optimal in[10]
2. If τ^* is an optimal stopping time in [10], then $\tau_0^N \leq \tau^*$ $P.a.s$
3. The sequence $(S_k^N)_{0\leq k\leq N}$ is the smallest super martingale which dominates $(G_k)_{n\leq k\leq N}$.
4. The stopped sequence $(S_{k\wedge\tau_n^N}^N)$ is a martingale.

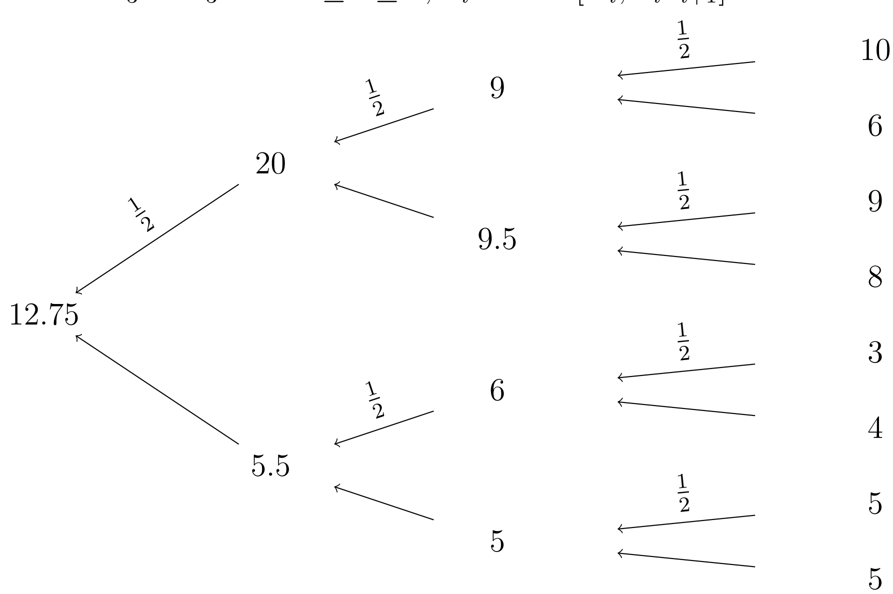
Detail proof can be found in [22].

Binomial Tree Example:

Suppose we have a binomial tree for G_i for $0 \leq i \leq 3$



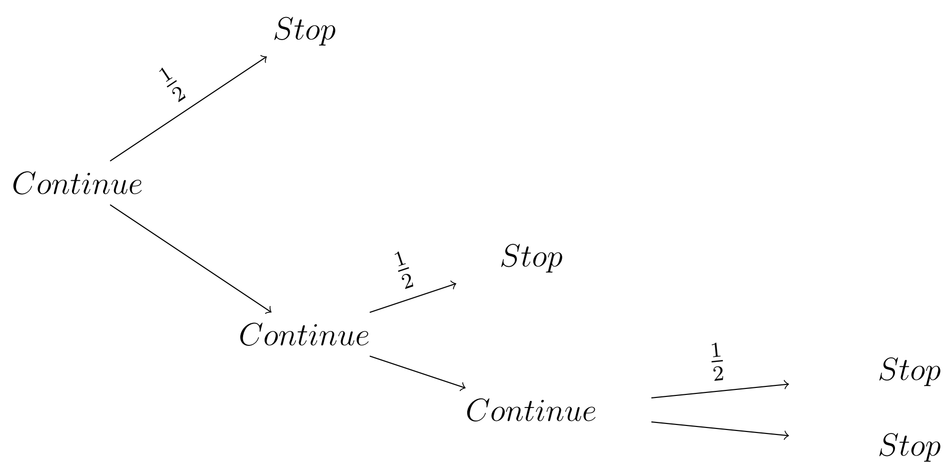
Then we can use the Bellman equation to get the value of V_i for $0 \leq i \leq 3$, which starts $i = 3$ and let $V_3 = G_3$. For $0 \leq i \leq 2$, $V_i = \max\left[G_i, \mathbb{E}_i V_{i+1}\right]$.



After this, we can have the optimal stopping time:

$$\inf\left\{0 \leq i \leq 3 \,|\, V_i = G_i\right\}$$

The decision tree is as follows:



Continuous Case:

Suppose $X = (X)_{0 \leq t < \infty}$ is a strong Markov process with continuous paths in the probability space $(\Omega, (\mathcal{F})_{t \geq 0}, \mathcal{F}, P_x)$. In addition, we assume X takes values in a measurable space $(R^d, \mathcal{B}(R^d))$, which starts at x under P_x for $x \in R^d$. Moreover, $(\mathcal{F}_{t \geq 0})$ satisfies the usual condition.

Definition 2.7 A random variable $\tau : \Omega \to [0, \infty]$ is called Markov time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. A Markov time is called a stopping time if $\tau < \infty$ $P.a.s.$The family of all stopping times will be denoted by M.

Definition 2.8 For $0 \leq t \leq T < \infty$, Define $M_t^T = \{\tau \in M : t \leq \tau \leq T\}$

Assumption 2.9 Gain function $G : R^d \to R$ is Borel measurable function satisfying:

$$\mathbb{E}_x(\sup_{0 \leq t \leq T} |G(X_t)|) < \infty \text{ and } G(X_\infty) = 0 \text{ P.a.s. for all } x \in R^d.$$

where, T is a fixed number in $\overline{R^+}$.

Based on this assumption, we can get that $E_x |G(X_\tau)| < \infty$ and $\liminf_{t\to\infty} E_x I(\tau > t) |G(X_\tau)| < \infty$ for all $x \in R^d$ and stopping times τ. However, this assumption does not hold for some functions and processes. If this assumption does not hold, we can prove all theories are still true as long as the optimal stopping time in the set: $\Psi = \left\{\tau : \forall x \in R^d, \mathbb{E}_x G(X_\tau) < \infty, \liminf_{t\to\infty} \mathbb{E}_x I(\tau > t) |G(X_\tau)| < \infty\right\}$

(time independent)optimal stopping problem is:

$$V(x) = \sup_{0\leq\tau\leq T} E_x G(X_\tau) \tag{13}$$

where τ is a stopping time with respect to $(\mathcal{F}_{t\geq 0})$ and $T \in \overline{R^+}$.

(time dependent)optimal stopping problem:

$$V(t,x) = \operatorname{ess\,sup}_{t\leq\tau\leq T} E_{(t,x)} G(\tau, X_\tau) \tag{14}$$

where τ is a stopping time with respect to $(\mathcal{F}_{t\geq 0})$ and $T \in \overline{R^+}$.It is well know that the above equation is called snell envelope.

There are a few natural questions that arise at this point before we are going to solve the main problem:

1. Which decision we should make? Stop or continue?
2. If we choose to continue, how to find the optimal stopping time?

Let's try to solve the first question. If current value $G(x) \geq E_x G(X_\tau)$ for all stopping time τ, then we should choose to stop. Otherwise we will tend to lose value. On the other hand, if there exists a stopping time α such that $G(x) < E_x G(X_\alpha)$, then we should choose to continue because we can find at least one strategy to get more value.

Now, let's think of the second question. It's not difficult to get the following results. If X is time-homogeneous Markov process and optimal stopping problem is infinite case, then the continuation and stop region if exists does not change over time and is independent with the state variable. This is simply because we are actually facing a same question as time goes. Hence, we just need to find the optimal constant boundary in this case. For example, the boundary of the perpetual American put is constant if we assume the underlying asset follows geometric Brownian motion. However, if X is time-inhomogeneous Markov process or optimal stopping problem is finite case, then the continuation and stop region change over time or depend on the state variable. For example, the boundary of the finite American put is a function of time if we assume the underlying asset follows geometric Brownian motion. If the underlying asset does not follow geometric Brownian motion, then the boundary may be a function of both time and state variable.

Following trivial cases are easy to get by using optional sampling theorem. If $\{G(X_t)\}_{0\leq t\leq T}$ is sub martingale under P_x, then $\tau = T$. If $\{G(X_t)\}_{0\leq t\leq T}$ is super martingale under P_x, then $\tau = 0$. However, what's the optimal stopping time if $\{G(X_t)\}_{0\leq t\leq T}$ is neither sub martingale nor supermartingale? Generally, the optimal stopping time should be $\tau = inf\,\{t \geq 0 : X_t \in D\}$, where $D = \{x : V(x) = G(x)\}$. $V(x)$ represents the maximum possible value given time and state variable x. (Note: here x includes the time dimension.) Then the key thing is to find $V(x)$.

Let me use the following example to show the importance of the assumption of uniformly integrability.

Example 2.8 Consider the following optimal stopping problem:

$$V(x) = \sup_{0 \leq \tau \leq \infty} E_x(B_\tau - arctan(\tau))$$

It's easy to see the gain function is supermartingale, then optimal stopping time $\tau = 0$. Therefore, we can get $V(x) = x$. However, define $\tau^* = inf\{t \geq 0 : B_t = 2x + 1\}$, then $E_x(B_{\tau^*} - arctan(\tau^*)) \geq 2x$. This means $\mathbb{E}_x(B_{\tau^*} - arctan(\tau^*)) > V(x)$. Contradict with the theorem.

Theorem 2.9 Suppose $\widehat{V}$ is the smallest super harmonic function which dominates the gain function G on R^d. In addition assuming that $\widehat{V}$ is lsc and G is usc. Set $D = \left\{x \in R^d : \widehat{V}(x) = G(x)\right\}$ and $\tau_D = inf\{t \geq 0 : X_t \in D\}$.

Then:

If $P_x(\tau_D < \infty) = 1$ for all $x \in R^d$, then $\widehat{V} = V$ and τ_D is optimal.

If $P_x(\tau_D < \infty) < 1$ for some $x \in R^d$, then there is no optimal stopping time.

Detail proof can be found in[22].

2.2 Verification Theorem

Theorem 2.10 Value Function Independent on Time

Suppose there exists a measurable function $\widehat{V}(x) : R^d \to R$ satisfying

1. $\widehat{V}(x) \geq G(x)$ for all $x \in R^d$.
2. $\widehat{V}(x)$ is super harmonic function w.r.t $(X)_{t\geq 0}$.

3. There exists a stopping time $\varsigma \in M_0^T$ such that

$$\widehat{V}(x) = E_x G(X_\varsigma)$$

for all $x \in R^d$.

Then we can have:

1. $\widehat{V}(x) = V(x)$ for any $x \in R^d$.
2. If $\widehat{V}$ is lsc and $G(x)$ is usc, then $\tau^* = inf \left\{ t \geq 0 : \widehat{V}(X_t) = G(X_t) \right\}$ is the smallest optimal stopping time.

Proof: Because $\widehat{V}(x) \geq G(x)$ for all $x \in R^d$,it's easy to see that $E_x G(X_\tau) \leq E_x \widehat{V}(X_\tau)$. By the definition of super harmonic function in the continuous time, we can conclude that $E_x G(X_\tau) \leq \widehat{V}(x)$ for any stopping time τ and $x \in R^d$. Therefore $\sup_{0 \leq \tau < \infty} E_x G(X_\tau) \leq \widehat{V}(x)$ for any $x \in R^d$. Because of the third property, we can conclude that $\widehat{V}(x) = V(x)$. In order to prove τ^* is the smallest optimal time, we first claim that For any optimal stopping time τ, $\widehat{V}(X_\tau) = G(X_\tau)$ $P.a.s$. This is true otherwise there exists an optimal stopping time such that $P_x(\widehat{V}(X_\tau) > G(X_\tau)) > 0$. Hence, $E_x G(X_\tau) < E_x \widehat{V}(X_\tau) \leq \widehat{V}(x)$ which contradicts with the assumption that τ is optimal. Moreover, because $\widehat{V}(x)$ is lsc and $G(x)$ is usc, τ^* is a stopping. Hence, τ^* is the smallest optimal stopping time.

From the theorem above, we can get the following result: Suppose X_t be a d-dimensional process satisfying the setup and does not include time dimension.(For example, d-dimensional Ito diffusion process). In addition, let the gain function $G(x)$ satisfy assumption 2.9 w.r.t $X = (X_t)_{0 \leq t < \infty}$ and x=L is the global maximum point of

$G(x)$. Then $\widehat{V}(x) = G(L)$ and $\tau = inf\,\{t \geq 0 : X_t = L\}$ is an optimal stopping time if $\tau < \infty$ P-a.s.

Theorem 2.11 Value Function Dependent on Time

Suppose there exists a measurable function $\widehat{V}(t,x) : R^+ \otimes R^d \to R$ satisfying

1. $\widehat{V}(t,x) \geq G(t,x)$ for all $(t,x) \in R^+ \otimes \in R^d$.
2. $\widehat{V}(t,x)$ is super harmonic function w.r.t $(t, X_t)_{t \geq 0}$.
3. For any $t \geq 0$, there exists stopping times $\varsigma_t \in M_t^T$ such that

$$\widehat{V}(t,x) = E_{(t,x)} G(\varsigma_t, X_{\varsigma_t})$$

for all $(t,x) \in R^+ \otimes R^d$.

Then we can have:

1. $\widehat{V}(t,x) = V(t,x)$ for $(t,x) \in R^+ \otimes R^d$.
2. If $\widehat{V}(t,x)$ is lsc and $G(t,x)$ is usc, then $\tau^* = inf\left\{t \geq 0 : \widehat{V}(t, X_t) = G(t, X_t)\right\}$ is the smallest optimal stopping time.

Proof: Because $\widehat{V}(t,x) \geq G(t,x)$ for all $(t,x) \in R^+ \otimes \in R^d$, it's easy to see that $\mathbb{E}_(t,x) G(\tau, X_\tau) \leq \mathbb{E}_{(t,x)} \widehat{V}(\tau, X_\tau)$. By the definition of superharmonic function in the continuous time, we can conclude that $\mathbb{E}_{(t,x)} G(\tau, X_\tau) \leq \widehat{V}(t,x)$ for any stopping time $\tau \in M_t^T$ and $(t,x) \in R^+ \otimes \in R^d$. Therefore $\sup_{t \leq \tau \leq T} \mathbb{E}_{(t,x)} G(\tau, X_\tau) \leq \widehat{V}(t,x)$ for any $(t,x) \in R^+ \otimes \in R^d$. Because of the third property, we can conclude that $\widehat{V}(t,x) = V(t,x)$ for all $(t,x) \in R^+ \otimes \in R^d$. In order to prove τ^* is the smallest optimal time, we first claim that $\widehat{V}(\tau, X_\tau) = G(\tau, X_\tau)\ P - a.s.$ for any optimal stopping time τ.

If it's not true, then $P_x(\widehat{V}(\tau, X_\tau) > G(\tau, X_\tau)) > 0$. Hence, $\mathbb{E}_x G(\tau, X_\tau) < E_x \widehat{V}(\tau, X_\tau) \leq \widehat{V}(t, x)$ which contradicts with the assumption that τ is an optimal stopping time. Moreover, because $\widehat{V}(t, x)$ is lsc and $G(t, x)$ is usc, τ^* is a stopping. Hence, τ^* is the smallest optimal stopping time.

The procedure to apply this theorem is first to guess the stopping time ς_t (For example, the first hitting time to the constant bound). Then calculate $\widehat{V}(t, x) = E_(t, x)G(\varsigma_t, X_{\varsigma_t})$. If the function $\widehat{V}(t, x)$ satisfies the first two properties in the above theorem , then we can conclude that ς_t are optimal stopping times for the snell envelope and $\widehat{V}(t, x) = V(t, x)$ for all $(t, x) \in R^+ \otimes R^d$. As we said before, bound is constant for infinite time horizon and time-homogeneous Markov process but it will depend on time for finite time horizon and time-homogeneous Markov process . Therefore, we will assume bound is $b(t, x)$ instead of constant b. The first thing is to calculate $\widehat{V}(x) = \mathbb{E}_x G(X_\varsigma)$ for a very complexed function $G(x)$. In order to calculate the expected value, one way is to use Laplace transformation. We will give some examples in section 5. Another way is to transform the problem to the boundary value problems. Examples will be in the appendix.

2.3 Two Stopping Times

In this section, we remain the assumption of X and G from the previous section. Consider the (time independent) optimal stopping problem:

$$V(x) = \sup_{0 \leq \xi \leq \varsigma \leq T} E_x[G_1(X_\xi) + G_2(X_\varsigma)] \tag{15}$$

where ξ and ς are stopping times with respect to $(\mathcal{F}_{t \geq 0})$ and $T \in \overline{R^+}$. Consider the (time dependent) optimal stopping problem:

$$V(t,x) = \operatorname{ess\,sup}_{t\leq\xi\leq\varsigma\leq T} \mathbb{E}_{(t,x)}[G_1(\xi, X_\xi) + G_2(\varsigma_t, X_{\varsigma_t})] \tag{16}$$

where ξ and ς are stopping times with respect to $(\mathcal{F}_{t\geq 0})$ and $T \in \overline{R^+}$.

Theorem 2.12 Value Function Independent on Time

Suppose $G_1(x)$ and $G_2(x) : R^d \to R$ are continuous functions. If there exists measurable functions $\widehat{U}(x)$ and $\widehat{V}(x) : R^d \to R$ satisfying:

1. $\widehat{U}(x) \geq G_2(x)$ for all $x \in R^d$.
2. $\widehat{U}(x)$ is super harmonic function w.r.t $(X)_{t\geq 0}$.
3. There exists a stopping time ς such that

   $$\widehat{U}(x) = \mathbb{E}_x G_2(X_\varsigma)$$

   for all $x \in R^d$.
4. $\widehat{V}(x) \geq G_1(x) + \widehat{U}(x)$ for all $x \in R^d$.
5. $\widehat{V}(x)$ is a superharmonic function w.r.t $(X)_{t\geq 0}$.
6. There exists a stopping time ξ such that

   $$\widehat{V}(x) = \mathbb{E}_x[G_1(X_\xi) + \widehat{U}(X_\xi)]$$

   for all $x \in R^d$.

Then

1. $\widehat{V}(x) = V(x)$.

2. If $\widehat{V}(x)$ and $\widehat{U}(x)$ are continuous functions, then

$$\xi^* = inf \left\{ t \geq 0 : \widehat{V}(X_t) = (G_1 + \widehat{U})(X_t) \right\}$$

and

$$\varsigma^* = inf \left\{ t \geq \xi^* : \widehat{U}(X_t) = G_2(X_t) \right\}$$

are a pair of the stopping time (ξ^*, ς^*)

According to the theorem 2.12, we can get for any $t \leq \tau_1 \leq \tau_2 \leq T$,

$$\widehat{U}(X_{\tau_1}) \geq \mathbb{E}_{X_{\tau_1}} G_2(X_{\tau 2})$$

and

$$\widehat{V}(x) \geq \mathbb{E}_x[G_1(X_{\tau 1}) + \widehat{U}(X_{\tau_1})]$$

Therefore we can get:

$$\begin{aligned} \widehat{V}(x) &\geq \mathbb{E}_x[G_1(X_{\tau 1}) + \widehat{U}(X_{\tau_1})] \\ &\geq \mathbb{E}_x[G_1(X_{\tau 1}) + \mathbb{E}_{X_{\tau 1}} G_2(X_{\tau 2})] \\ &\geq \mathbb{E}_x[G_1(X_{\tau 1}) + G_2(X_{\tau 2})] \end{aligned}$$

On the other hand, because

$$\widehat{V}(x) = \mathbb{E}_x[G_1(X_\xi) + G_2(X_\varsigma)] = \mathbb{E}_x[G_1(X_{\xi^*}) + G_2(X_{\varsigma^*})]$$

Therefore

$$\widehat{V}(x) = V(x)$$

Moreover, if $\widehat{V}(x)$ and $\widehat{U}(x)$ are continuous functions, then (ξ^*, ς^*) are a pair of optimal stopping times. $\diamond$

Let X_t be a d-dimensional process satisfying the setup and does not include time dimension.(For example, d-dimensional Ito diffusion process). In addition, let the gain function $G_1(x)$ and $G_2(x)$ satisfy assumption 2.9 w.r.t $X = (X_t)_{0 \le t < \infty}$ and $x = L_1$ is the global maximum point of $G_1(x)$ and $x = L_2$ is the global maximum point of $G_2(x)$. Then $\widehat{V}(x) = G_1(L_1) + G_2(L_2)$, $\xi^* = inf\,\{t \ge 0 : X_t = L_1\}$ and $\tau^* = inf\,\{t \ge \xi : X_t = L_2\}$ are a pair of optimal stopping time if $\tau^*, \xi^* < \infty$ P-a.s.

Theorem 2.13 Value Function Dependent on Time

Suppose $G_1(t,x)$ and $G_2(t,x) : R^+ \otimes R^d \to R$ are continuous functions. If there exists measurable functions $\widehat{V}(t,x)$,$\widehat{U}(t,x) : R^+ \otimes R^d \to R$ satisfying

1. $\widehat{U}(t,x) \ge G_2(t,x)$ for all $(t,x) \in R^+ \otimes R^d$.
2. $\widehat{U}(t,x)$ is a superharmonic function w.r.t $(t, X_t)_{t \ge 0}$.
3. There exists stopping times $\varsigma_t \in_t^T$ such that

$$\widehat{U}(t,x) = E_{(t,x)} G_2(\varsigma_t, X_{\varsigma_t})$$

for all $(t,x) \in R^+ \otimes R^d$.

4. $\widehat{V}(t,x) \ge G_1(t,x) + \widehat{U}(t,x)$ for all $(t,x) \in R^+ \otimes R^d$.
5. $\widehat{V}(t,x)$ is superharmonic function w.r.t $(t, X_t)_{t \ge 0}$.

6. There exists stopping times $\xi_t \in M_t^T$ such that

$$\widehat{V}(t,x) = E_{(t,x)}(G_1 + \widehat{U})(\xi_t, X_{\xi_t})$$

for all $(t,x) \in R^+ \otimes R^d$.

1. $\widehat{V}(t,x) = V(t,x)$.

2. If $\widehat{V}$ and $\widehat{U}$ are continuous functions, then

$$\xi_t^* = \inf\left\{s \geq t : \widehat{V}(t,X_t) = (G_1 + \widehat{U})(t,X_t)\right\}$$

and

$$\varsigma_t^* = \inf\left\{t \geq \xi_t^* : \widehat{U}(t,X_t) = G_2(t,X_t)\right\}$$

are a pair of the optimal stopping times.

According to the theorem 2.12, we can get for any $t \leq \tau_1 \leq \tau_2 \leq T$,

$$\widehat{U}(\tau_1, X_{\tau_1}) \geq \mathbb{E}_{(\tau_1, X_{\tau_1})} G_2(\tau_2, X_{\tau 2})$$

and

$$\widehat{V}(t,x) \geq \mathbb{E}_{(t,x)}[G_1(\tau_1, X_{\tau 1}) + \widehat{U}(\tau_1, X_{\tau_1})]$$

Therefore we can get:

$$\begin{aligned}\widehat{V}(t,x) &\geq \mathbb{E}_{(t,x)}[G_1(\tau_1, X_{\tau 1}) + \widehat{U}(\tau_1, X_{\tau_1})] \\ &\geq \mathbb{E}_{(t,x)}[G_1(\tau_1, X_{\tau 1}) + \mathbb{E}_{(\tau_1, X_{\tau 1})} G_2(\tau_2, X_{\tau 2})] \\ &\geq \mathbb{E}_{(t,x)}[G_1(\tau_1, X_{\tau 1}) + G_2(\tau_2, X_{\tau 2})]\end{aligned}$$

On the other hand, because

$$\widehat{V}(t,x) = \mathbb{E}_{(t,x)}[G_1(\xi_t, X_{\xi_t}) + G_2(\varsigma_t, X_{\varsigma_t})] = \mathbb{E}_{(t,x)}[G_1(\xi_t^*, X_{\xi_t^*}) + G_2(\varsigma_t^*, X_{\varsigma_t^*})]$$

Therefore

$$\widehat{V}(t,x) = V(t,x)$$

Moreover, if $\widehat{V}(t,x)$ and $\widehat{U}(t,x)$ are continuous functions, then (ξ_t^*, ς_t^*) are a pair of optimal stopping times. $\diamond$

2.4 Examples

Drifted Brownian Motion:

In this section, we assume the process $(X_t)_{0 \leq t < \infty}$ satisfying:

$$dX_t = \mu t + dW_t$$

with nonrandom initial value X_0 where $(W_t)_{0 \leq t < \infty}$ is 1-dimensional Brownian motion and μ is a constant.

The optimal stopping problem is:

$$V(t,x) = \operatorname{ess\,sup}_{t\leq\tau<\infty} \mathbb{E}_{(t,x)} e^{-r\tau}\tilde{G}(X_\tau)$$

where $e^{-rt}\tilde{G}(x)$ satisfies assumption 2.9 w.r.t $(X_t)_{0\leq t<\infty}$.

In the case of $\mu \geq 0$, let stopping times ς_t be the first hitting time to a constant bound $L > \max(x,0)$, i.e $\varsigma_t = inf\,\{s \geq t : X_s = L\}$ and Then we have:

$$\begin{aligned} g(t,x) &= E_{(t,x)} e^{-r\varsigma_t}\tilde{G}(X_{\varsigma_t}) \\ &= \tilde{G}(L) E_{(t,x)} e^{-r\varsigma_t} \\ &= \tilde{G}(L) e^{-rt} E_{(0,x)} e^{-r\varsigma_0} \end{aligned}$$

By using the Laplace transform for the first passage time of drifted Brownian motion, we can get:

$$g(t,x) = \tilde{G}(L) e^{-rt} e^{(x-L)(-\mu+\sqrt{\mu^2+2r})} \tag{17}$$

In the case of $\mu < 0$, let stopping times ς_t be the first hitting time to a constant bound $L < \min(x,0)$, i.e $\varsigma_t = inf\,\{s \geq t : X_s = L\}$ and Then we have:

$$\begin{aligned} g(t,x) &= \mathbb{E}_{(t,x)} e^{-r\varsigma_t}\tilde{G}(X_{\varsigma_t}) \\ &= \tilde{G}(L) E_{(t,x)} e^{-r\varsigma_t} \\ &= \tilde{G}(L) e^{-rt} E_{(0,x)} e^{-r\varsigma_0} \end{aligned}$$

By using the Laplace transform for the first passage time of drifted Brownian motion, we can get:

$$g(t,x) = \tilde{G}(L)e^{-rt}e^{(L-x)(\mu+\sqrt{\mu^2+2r})} \tag{18}$$

According to the theorem in the optimal stopping chapter, we can get the following propositions.

Proposition 2.15 For $\mu \geq 0$, Suppose $\tilde{G} \in C^2(R)$ satisfies following conditions for some L.

1. $\tilde{G}'(L_+) = \tilde{G}(L)(-\mu + \sqrt{\mu^2 + 2r})$.(Smooth Pasting Condition)
2. $(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\tilde{G}(x) \leq 0$ for $x > L$.(Super harmonic Condition)
3. $\tilde{G}(L)e^{L(\mu-\sqrt{\mu^2+2r})} \geq \tilde{G}(x)e^{x(\mu-\sqrt{\mu^2+2r})}$, for all $x < L$.(Dominating Condition)

then:

$$V(t,x) = \begin{cases} \tilde{G}(L)e^{-rt}e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}, & for \quad x < L \\ e^{-rt}\tilde{G}(x), & for \quad x \geq L \end{cases}$$

and $\tau = \inf\{t \geq 0 : X_t \in [L, \infty)\}$ is an optimal stopping time. Define:

$$\widehat{V}(t,x) = \begin{cases} \tilde{G}(L)e^{-rt}e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}, & for \quad x < L \\ e^{-rt}\tilde{G}(x), & for \quad x \geq L \end{cases}$$

and

$$\tau_t = \inf\{s \geq t : X_s \in [L, \infty)\}$$

Since $\mu \geq 0$, then we can see τ is a stopping time. According to equation (19), we can have:

$$\widehat{V}(t,x) = \mathbb{E}_{(t,x)} e^{-r\tau_t} \tag{19}$$

Because of the dominating condition, we get for $x < L$:

$$\tilde{G}(L) e^{-rt} e^{(x-L)(-\mu+\sqrt{\mu^2+2r})} \geq e^{-rt} \tilde{G}(x) \tag{20}$$

Because for $x < L$, we have

$$(-r + \mu \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2})[\tilde{G}(L) e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}] = 0 \tag{21}$$

According to superharmonic property, we can have for $x > L$:

$$(-r + \mu \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2}) \tilde{G}(x) \leq 0 \tag{22}$$

Because of the smooth pasting condition, we can see $\frac{\partial V(t,x)}{\partial x}$ exists at for all $(t, x) \in (R^+ \otimes R)$. Therefore, we have

$$V(t,x) = \begin{cases} \tilde{G}(L) e^{-rt} e^{(x-L)(-\mu+\sqrt{\mu^2+2r})}, & for \quad x < L \\ e^{-rt} \tilde{G}(x), & for \quad x \geq L \end{cases}$$

and $\tau = \inf \{t \geq 0 : X_t \in [L, \infty)\}$ is an optimal stopping time. $\diamond$

Proposition 2.16 For $\mu \leq 0$, suppose $\tilde{G} \in C^2(R)$ except for finite number of points satisfies following conditions :

1. $\tilde{G}^{'}(L_-) = \tilde{G}(L)(-\mu - \sqrt{\mu^2 + 2r})$. (Smooth Pasting Condition)

2. $(-r+\mu\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2})\tilde{G}(x) \leq 0$ for $x < L$.(Super harmonic Condition)

3. $\tilde{G}(L)e^{L(\mu+\sqrt{\mu^2+2r})} \geq \tilde{G}(x)e^{x(\mu+\sqrt{\mu^2+2r})}$, for all $x > L$.(Dominating Condition)

then

$$V(t,x) = \begin{cases} e^{-rt}\tilde{G}(x), & for \quad x \leq L \\ \tilde{G}(L)e^{-rt}e^{(L-x)(\mu+\sqrt{\mu^2+2r})}, & for \quad x > L \end{cases}$$

and $\tau = \inf\{t \geq 0 : X_t \in (-\infty, L]\}$ is an optimal stopping time.

proof will be very similar with the previous one.

Example

Recall drift Brownian motion X_t with $\mu = 0$, $X_0 = x$. $G(x) = bx$, where b is a negative constant.

$$V(x) = \sup_{0 \leq \tau < \infty} E_x[e^{-r\tau}bX_\tau]$$

Because

$$d[e^{-rt}bX_t] = (-rbX_t)e^{-rt}dt + be^{-rt}dW_t$$

then if $X_t > 0$, drift part is greater than 0 and we should chose to continue. Let's guess the stopping region is:

$$\tau = inf\{t \geq 0 : X_t \in (-\infty, L]\}$$

Using the smooth pasting, we can get:

$$L = -\frac{1}{\sqrt{2r}}$$

After checking the dominating property and superharmonic property, we can conclude

$\tau = inf\left\{t \geq 0 : X_t \in (-\infty, -\frac{1}{\sqrt{2r}}]\right\}$ is the optimal stopping time and

$$V(x) = \begin{cases} e^{-rt}bx, & for \quad x \leq -\dfrac{1}{\sqrt{2r}} \\ -e^{-rt}\dfrac{b}{\sqrt{2r}}e^{(-\frac{1}{\sqrt{2r}}-x)\sqrt{2r}}, & for \quad x > -\dfrac{1}{\sqrt{2r}} \end{cases}$$

Geometric Brownian Motion:

In this part, we assume the process $(X_t)_{0\leq t<\infty}$ satisfying:

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

where, $(W_t)_{0\leq t<\infty}$ is 1-dimensional Brownian motion and μ and σ are constants.

The optimal stopping problem is:

$$V(x) = \sup_{0\leq\tau<\infty} E_x e^{-r\tau}\tilde{G}(X_\tau)$$

where,$e^{-rt}\tilde{G}(x)$ satisfies assumption 1.1 w.r.t $(X_t)_{0\leq t<\infty}$.

Let's first think of a special stopping time which is the first hitting time to a constant bound L, i.e $\tau_L = \inf\{t \geq 0 : X_t = L\}$ and $\tau_L^t = inf\{s \geq t : X_t = L\}$Then we have:

$$\begin{aligned} g(t,x) &= E_{(t,x)}e^{-r(\tau_L^t)}\tilde{G}(X_{\tau_L^t}) \\ &= \tilde{G}(L)E_{(t,x)}e^{-r\tau_L^t} \\ &= \tilde{G}(L)e^{-rt}E_{(0,x)}e^{-r\tau_L} \end{aligned} \tag{23}$$

By using the Laplace transform for the first passage time of geometric Brownian motion, we can get:

$$g(t,x) = \begin{cases} \tilde{G}(L)e^{-rt}(\frac{L}{x})^{\frac{\mu_1-\sqrt{\mu_1^2+2r}}{\sigma}}, & for \quad x < L \\ \tilde{G}(L)e^{-rt}(\frac{L}{x})^{\frac{\mu_1+\sqrt{\mu_1^2+2r}}{\sigma}}, & for \quad x \geq L \end{cases} \tag{24}$$

where $\mu_1 = \frac{\mu}{\sigma} - \frac{1}{2}\sigma$. According to the theorem 1.5, we can get the following propositions.

Proposition 2.18 Suppose $\mu - \frac{1}{2}\sigma^2 \geq 0$.If $G \in C^2(R)$ satisfies following condition for some L:

1. $\tilde{G}'(L_+) = \tilde{G}(L)\frac{-\mu_1+\sqrt{\mu_1^2+2r}}{\sigma L}$.(Smooth Pasting Condition)
2. $(-r + \mu x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2})\tilde{G}(x) \leq 0$ for $x > L$.(Super harmonic Condition)
3. $\tilde{G}(L)L^{\frac{\mu_1-\sqrt{\mu_1^2+2r}}{\sigma}} \geq \tilde{G}(x)x^{\frac{\mu_1-\sqrt{\mu_1^2+2r}}{\sigma}}$, for all $x < L$.(Dominating Condition)

then

$$\widehat{V}(t,x) = \begin{cases} \tilde{G}(L)e^{-rt}(\frac{L}{x})^{\frac{\mu_1-\sqrt{\mu_1^2+2r}}{\sigma}}, & for \quad x < L \\ e^{-rt}\tilde{G}(x), & for \quad x \geq L \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in [L, \infty)\}$ is an optimal stopping time. The proof will be same as before.

Proposition 2.19 Suppose $\mu - \frac{1}{2}\sigma^2 \leq 0$. If $G \in C^2(R)$ satisfies following condition for some L:

1. $\tilde{G}'(L_-) = \tilde{G}(L)\frac{\mu_1+\sqrt{\mu_1^2+2r}}{\sigma L}$. (Smooth Pasting Condition)
2. $(-r + \mu x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2})\tilde{G}(x) \leq 0$ for $x < L$.(Super harmonic Condition)
3. $\tilde{G}(L)L^{\frac{\mu_1+\sqrt{\mu_1^2+2r}}{\sigma}} \geq \tilde{G}(x)x^{\frac{\mu_1+\sqrt{\mu_1^2+2r}}{\sigma}}$, for all $x > L$.(Dominating Condition)

then

$$V(t,x) = \begin{cases} e^{-rt}\tilde{G}(x), & for \quad x \leq L \\ \tilde{G}(L)e^{-rt}(\frac{L}{x})^{\frac{\mu_1+\sqrt{\mu_1^2+2r}}{\sigma}}, & for \quad x > L \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in (-\infty, L]\}$ is an optimal stopping time.

CHAPTER 3: AMERICAN OPTION MARKET

In this chapter, we will extend HJM approach to American option market by using theorems in the optimal stopping problems. As we will see later, there are a number of differences between European options and American options. First, there is no optimal stopping time in European option market but it is a very important concept for American options. Second, how to model volatility is the key issue for European options. However, we will show how to model drift is the key issue for American options. Our focus will be about how to build an arbitrage free model for the drift.

In this chapter, we will give the HJM drift condition. In addition, as counterpart to the forward rate for bond market, the forward implied volatilities for European option market, here we introduce the forward drift for American option. Also, we introduce forward optimal stopping rule as counterpart to the classic stopping rule.

We will start with model setup in section one. In section two, we will give the necessary drift condition. We will introduce the forward stopping rule in section three. In section four, we will give the sufficient drift condition. In section five, we will discuss the relationship between spot model and forward model.

Given probability space $(\Omega, P, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F})$, where $\mathcal{F}_t$ is the natural filtration for the multi-dimensional Brownian motion W_t. Fix a finite time horizon T.

Let us consider the following problem. We define G_t:

$$dG_t = \mu_t dt + \sigma_t dW_t$$

with initial value $G(0)$. μ_t and σ_t are adapted processes satisfying the condition that G_t has unique strong solution.

Assumption 3.1

$$\mathbb{E}(\sup_{0\leq t\leq T}|G_t|)<\infty$$

Recall from the previous chapter, if we can find stopping times $\tau_t \in M_t^T$ for $t \geq 0$ such that $\widehat{V}_t = \mathbb{E}_t G_{\tau_t}$ satisfying it is a super martingale dominating G_t, then $\widehat{V}_t$ is the snell envelop for G_t.

For any stopping times $\{\tau_t\}_{0\leq t\leq T}$ such that $\tau_t \in M_t^T$, according to Hunt's stopping time theorem, we can get:

$$\begin{aligned}
\mathbb{E}_t G_{\tau_t} &= G_t + \mathbb{E}_t \int_t^{\tau_t} \mu_u du + \mathbb{E}_t \int_t^{\tau_t} \sigma_u dW_u \\
&= G_t + \mathbb{E}_t \int_t^{\tau_t} \mu_u du \\
&= G_t + \mathbb{E}_t \int_t^{T} \mu_u 1(\tau_t \geq u) du \\
&= G_t + \int_t^{T} \mathbb{E}_t[\mu_u 1(\tau_t \geq u)] du.
\end{aligned}$$

Here, we can see that the value $\mathbb{E}_t G_{\tau_t}$ does not depend on the volatility of the underlying asset. In order to find $\mathbb{E}_t G_{\tau_t}$, we can assume that $\sigma_u = 0$ for $0 \leq u \leq T$, and thus $G(t)$ satisfies:

$$dG_t = \mu_t dt \tag{25}$$

with initial value G_0. As we can see from the result above, from the point of view of the optimal stopping problem, μ_t plays a very important role.

3.1 Model Setup

Definition 3.2 Recall the definition of G_t with no volatility part:

$$dG_t = \mu_t dt \tag{26}$$

with initial value G_0.

Notation:

1. $V(0,T) = \sup_{0 \le \tau \le T} \mathbb{E} G_\tau$
2. $V(t,T) = \operatorname{ess\,sup}_{t \le \tau \le T} \mathbb{E}_t G_\tau$
3. $\tau_t = \inf \{t \le s \le T | V(s,T) = G(s)\}$

With the above notation, we have

$$V(t,T) = \mathbb{E}_t G_{\tau_t} = G_t + \int_t^T \mathbb{E}_t[\mu_u 1(\tau_t \ge u)]du.$$

As counterpart to the forward rate or forward implied volatility, here we introduce forward drift:

$$f_t(u) = \mathbb{E}_t[\mu_u 1(\tau_t \ge u)]. \tag{27}$$

Then for all $0 \le t \le T$,

$$f_t(t) = \mu_t. \tag{28}$$

This is the Spot Consistency Condition for forward drift. Recall the spot rate for forward rate is short rate for the fixed income market, and for implied forward volatility is spot volatility for European option market. Here we can see the spot rate for forward drift is spot drift for American option market. In addition, recall the definition of τ_0, we have $\tau_0 = \inf \left\{ 0 \leq t \leq T | \int_t^T f_t(u)du \leq 0 \right\}$. Here the forward problem for American option market is:

$$dG_t = \begin{cases} \mu_t dt, & G_0 \\ f_t(t) = \mu_t, & for \quad 0 \leq t \leq T \\ df_t(u) = \alpha_t(u)dt + \beta_t(u)dW_t, & f_0(u) \end{cases}$$

Then the question is "what is the Drift Condition given the initial value $G(0)$ and $f_0(u)$ for all $0 \leq u \leq T$"? We will show later that the Drift Condition here is described by admissible drift surface $\alpha_t^\beta(u)$ given the volatility surface $\beta_t(u)$.

In order that the model above is arbitrage free, we have

$$V(0,T) = G_0 + \int_0^T f_0(u)du \tag{29}$$

Otherwise, the market will have arbitrage opportunity.

3.2 Necessary Drift Condition

As we know $V(t,T)$ is a martingale in the continuous region $t \leq \tau_0$, we will use this property to derive the relation between α_t and β_t.

Theorem 3.3 Given initial value $f_0(u)$. Recall the definition τ_0, we can prove: for $0 \leq t \leq \tau_0$,

$$\int_t^T \alpha_t(u)du = 0.$$

Proof:

Let $z(t,T) = \int_t^T f_t(u)du$

$$\begin{aligned} dz(t,T) &= \int_t^T df_t(u)du - f_t(t)dt \\ &= \int_t^T [\alpha_t(u)dt + \beta_t(u)dW_t]du - f_t(t)dt \\ &= \int_t^T [\alpha_t(u)du - f_t(t)]dt + \int_t^T \beta_t(u)dudW_t \end{aligned}$$

Therefore

$$\begin{aligned} dV(t,T) &= dG_t + dz(t,T) \\ &= [\mu_t + \int_t^T \alpha_t(u)du - f_t(t)]dt + \int_t^T \beta_t(u)dudW_t \\ &= \int_t^T \alpha_t(u)dudt + \int_t^T \beta_t(u)dudW_t \end{aligned}$$

Thus for $0 \le t \le \tau_0$, we have

$$\int_t^T \alpha_t(u)du = 0.$$

The above theorem gives us the necessary condition for $\alpha_t(u)$ for $0 \le t \le \tau_0$. There are still two problems we have not solved here. First τ_0 is so far still unknown. The second problem is that we have not given any condition for the volatility surface $\beta_t(u)$ for $t \le u \le T$. We will give those results in the following subsection.

3.3 Forward Stopping Rule

In this section, we will introduce a forward approach to solve the classic optimal stopping problem. We will use this approach to find the optimal stopping time τ_0 and the value $V(t,T)$ before the stopping time τ_0 for the American option. We will not focus on the value function after the optimal stopping time.

Definition 3.4 Recall $f_0(u)$. Given adapted stochastic process $\alpha_t(u)$ and $\beta_t(u)$ for $t \leq u \leq T$, define

$$\overline{f}_t(u) = f_0(u) + \int_0^t \alpha_s(u)ds + \int_0^t \beta_s(u)dW_s$$

where $\alpha_t(u)$ and $\beta_t(u)$ satisfy the regular condition so that $\overline{f}_t(u)$ has a unique strong solution.

Definition 3.5

$$\widetilde{\tau}^* = \inf\left\{0 \leq t \leq T | \int_t^T \overline{f}_t(u)du \leq 0\right\}$$

In order to explain the forward stopping rule more clearly, we use the following binomial tree as an example to compare with the classic approach using backward induction.

Binomial Tree Example: Suppose $T = 3$.

In stead of modeling G_t, we model $\overline{f}_t(u)$. Note that $f_0(u)$ is observable in the market.

$$f_0(1)$$

$$\begin{tikzcd}
 & \overline{f}_{1H}(2) \\
f_0(2) \arrow[ur] \arrow[dr] & \\
 & \overline{f}_{1T}(2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \overline{f}_{2HH}(3) \\
 & \overline{f}_{1H}(3) \arrow[ur] \arrow[dr] & \\
 & & \overline{f}_{2HT}(3) \\
f_0(3) \arrow[uur] \arrow[ddr] & & \\
 & & \overline{f}_{2TH}(3) \\
 & \overline{f}_{1T}(3) \arrow[ur] \arrow[dr] & \\
 & & \overline{f}_{2TT}(3)
\end{tikzcd}$$

Then they will calculate the value of $U(i)$ for $0 \leq i \leq 3$, which starts at $i = 0$. Let $U(0) = \sum_{i=1}^{3} f_0(i)$. If $U(0) \leq 0$, we stop. Otherwise, we will continue and calculate $U(1) = \sum_{i=2}^{3} \overline{f}_1(i)$. If $U(1) \leq 0$, we stop. Otherwise, we will continue and calculate $U(2) = \overline{f}_2(3)$. The will have the decision tree

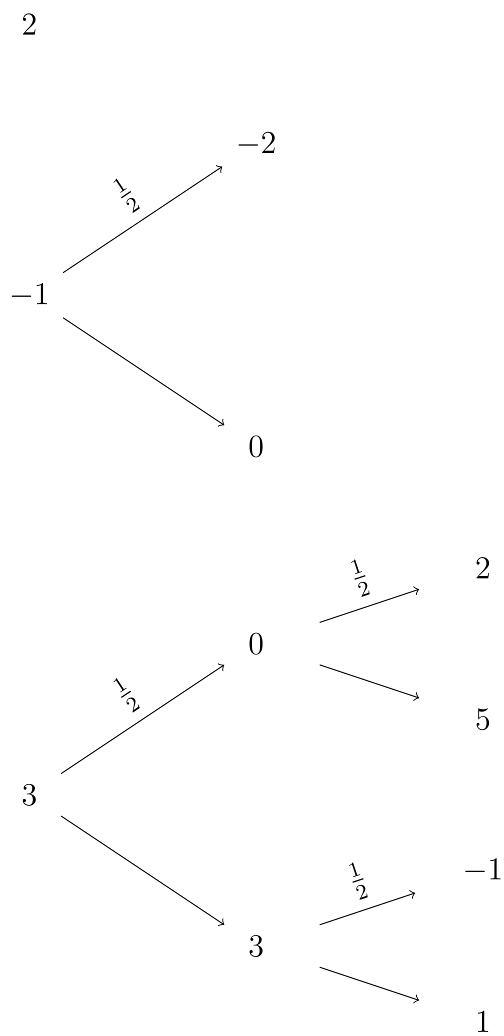
$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & U_{HH}(2) \\
 & U_H(1) \arrow[ur] \arrow[dr] & \\
 & & U_{HT}(2) \\
U(0) \arrow[uur] \arrow[ddr] & & \\
 & & U_{TH}(2) \\
 & U_T(1) \arrow[ur] \arrow[dr] & \\
 & & U_{TT}(2)
\end{tikzcd}
$$

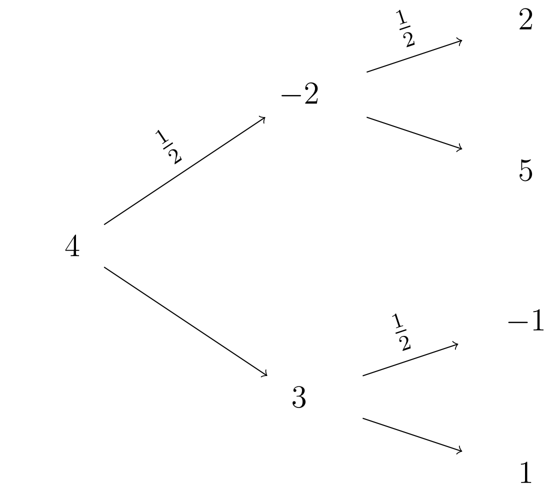
$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & Stop & \\
 & Continue \arrow[ur] \arrow[ddr] & & \\
 & & & Stop \\
 & & Continue \arrow[ur] \arrow[dr] & \\
 & & & Stop \\
Continue \arrow[uuuur] \arrow[dr] & & & \\
 & Stop & &
\end{tikzcd}
$$

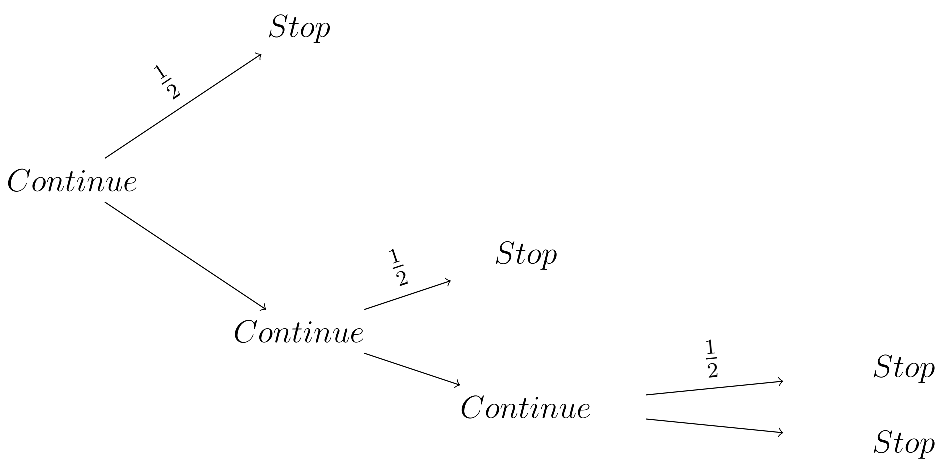
Now we give a numerical example to illustrate the concept presented above.

Example 3.6

Suppose initial value $f_0(u)$ and the modeled values $\overline{f}_t(u)$ are







Using the backward induction approach, one has to model the whole tree for G_t and calculate the value function $V(t, T)$ starting from the end period to decide the optimal stopping time. However, using forward decision approach, one does not need to sum over the whole tree. Consequently, the decision one makes on the optimal stopping time will depend on the more recent data and not the data far away.

Definition 3.7 Recall the definition of $\widetilde{\tau}^*$.

Given adapted process (volatility surface) $\{\beta_t(u)\}_{0\le t\le u\le T}$, we call $\{\alpha_t(u)\}_{0\le t\le u\le T}$ admissible drift surface if for $0 \le t \le \widetilde{\tau}^*$,

$$\int_t^T \alpha_t(u)du = 0.$$

We use the notation $\left\{\alpha_t^{\beta}(u)\right\}_{0\le t\le u\le T}$ to represent $\{\beta_t(u)\}_{0\le t\le u\le T}$ admissible drift surface.

It is easy to see that $\alpha_t(u) \equiv 0$ for all $0 \le t \le u \le T$ is admissible drift surface for any volatility surface $\beta_t(u)$. For the discrete-time case, it is not difficult to check whether the drift surface is admissible given the volatility surface. For the continuous-time case, it is easy to see that constant drift $\alpha_t(u) \equiv \alpha$ is not admissible for any volatility surface as long as $\alpha \ne 0$. We now give an example of constant volatility surface.

Given $\beta_t(u) \equiv \sigma$, we will check whether $\alpha_t(u)$ is $\beta_t(u)$ admissible or not. First, we need to calculate $\overline{f}_t(u)$. According to the definition,

$$\overline{f}_t(u) = f_0(u) + \int_0^t \alpha_s(u)ds + \sigma W_t.$$

Then we have

$$\int_t^T \overline{f}_t(u)du = \int_t^T f_0(u)du + \int_t^T \int_0^t \alpha_s(u)dsdu + \sigma W_t(T-t)$$

and

$$
\begin{aligned}
\widetilde{\tau}^{*} &= \inf\left\{0 \leq t \leq T | \int_t^T \overline{f}_t(u)du \leq 0\right\} \\
&= \inf\left\{0 \leq t \leq T | \int_t^T f_0(u)du + \int_t^T \int_0^t \alpha_s(u)dsdu + \sigma W_t(T-t) \leq 0\right\} \\
&= \inf\left\{0 \leq t \leq T | W_t \leq -\frac{\int_t^T f_0(u)du}{\sigma(T-t)} - \frac{\int_t^T \int_0^t \alpha_s(u)dsdu}{\sigma(T-t)}\right\}
\end{aligned}
$$

Then we need to check the drift condition:

$$
\int_t^T \alpha_t(u)du = 0 \quad \text{for } 0 \leq t \leq \widetilde{\tau}^{*}.
$$

3.4 Sufficient Drift Condition

Definition 3.9

Suppose $dX_t = \mu_t^x dt + \sigma_t^x dW_t$. If there exists a stopping time $\overline{\tau} \geq 0$ such that

$$
\mu_t^x \leq 0, \quad P \otimes dt - a.s.,
$$

for $t \geq \overline{\tau}$. Then we call process X_t a forward starting supermartingale and $\overline{\tau}$ is called the changing point for this process.

For any initial value G_0 and $f_0(u)$ for $0 \leq u \leq T$, we can always construct infinitely many forward starting supermartingales such that they are consistent with the initial values. These forward starting supermartingale will give us arbitrage free models.

Theorem 3.10 Given $G_0$and $f_0(u)$. Recall the definition of $V(0,T)$.

Recall the definition of (volatility surface) $\beta_t(u)$, its admissible (drift surface) $\alpha_t^{\beta}(u)$,

$\overline{f}_t(u)$ and $\widetilde{\tau}^*$. For the given (volatility surface) $\beta_t(u)$, construct a forward supermartingale

$$dX_t = \mu_t^x dt + \sigma_t^x dW_t$$

satisfying:

1. $X_0 = G_0$.
2. The changing point for X_t satisfies $\overline{\tau} = \widetilde{\tau}^*$.
3. $\mu_t^x = \overline{f}_t(t)$ for $0 \leq t \leq \overline{\tau}$.

Recall the definition of τ^* as the optimal stopping time for X_t i.e $\sup_{0\leq\tau\leq T}\mathbb{E}X_\tau = \mathbb{E}X_{\tau^*}$. Then

1. $\tau^* = \widetilde{\tau}^*$.
2. For $0 \leq t \leq \widetilde{\tau}^*$, $\operatorname{ess\,sup}_{t\leq\tau\leq T}\mathbb{E}_t X_\tau = X_t + \int_t^T \overline{f}_t(u)du$.
3. $V(0,T) = \sup_{0\leq\tau\leq T}\mathbb{E}X_\tau = \mathbb{E}X_{\widetilde{\tau}^*}$.

Proof. Define

$$\widehat{V}(t,T) = X_t + \int_t^T \overline{f}_t(u)du$$

and

$$U(t,T) = \operatorname{ess\,sup}_{t\leq\tau\leq T}\mathbb{E}_t X_\tau.$$

According to the definition of $\overline{f}_t(u)$ and admissible drift surface, we have for $0 \leq t \leq \widetilde{\tau}^*$:

$$d\widehat{V}(t,T) = [\sigma_t^x + \int_t^T \beta_t(u)du]dW_t \tag{30}$$

and

$$\widehat{V}(t,T) \geq X_t. \tag{31}$$

Because the changing point $\overline{\tau}$ of X_t satisfies $\overline{\tau} = \widetilde{\tau}^*$, we have $\tau^* \leq \widetilde{\tau}^*$. Therefore for $0 \leq t \leq \tau^*$,

$$\begin{aligned} U(t,T) &= \mathbb{E}_t X_{\tau^*} \\ &\leq \mathbb{E}_t \widehat{V}(\tau^*,T) \\ &= \widehat{V}(t,T). \end{aligned}$$

On the other hand, since for $0 \leq t \leq \tau^*$,

$$\widehat{V}(t,T) = \mathbb{E}_t \widehat{V}(\widetilde{\tau}^*,T) = \mathbb{E}_t X_{\widetilde{\tau}^*} \leq U(t,T). \tag{32}$$

According to equation above, we will get for $0 \leq t \leq \tau^*$,

$$U(t,T) = \widehat{V}(t,T). \tag{33}$$

Moreover, because X_t is continuous process, we will have

$$U(\tau^*,T) = X_{\tau^*}. \tag{34}$$

Then we have

$$\widehat{V}(\tau^*,T) = X_{\tau^*}. \tag{35}$$

Thus

$$\tau^* = \widetilde{\tau}^* \tag{36}$$

and

$$\sup_{0 \le \tau \le T} \mathbb{E}X_\tau = \mathbb{E}X_{\widetilde{\tau}^*} \tag{37}$$

By $\widehat{V}(0,T) = U(0,T)$ and $\widehat{V}(0,T) = V(0,T)$, we have

$$\mathbb{E}X_{\widetilde{\tau}^*} = X(0) + \int_t^T f_0(u)du = V(0,T). \tag{38}$$

$\diamond$

As we have seen from the previous subsection, $\alpha_t(u) \equiv 0$ for all $0 \le t \le u \le T$ is an admissible drift surface for any volatility surface $\beta_t(u)$, we can have the following corollary.

Corollary 3.11

Given adapted process (volatility surface) $\beta_t(u)$, define:

$$\overline{f}_t(u) = f_0(u) + \int_0^t \beta_s(u)dW_s$$

Construct a forward supermartingale $dX_t = \mu_t^x dt + \sigma_t^x dW_t$ satisfying:

1. $X_0 = G_0$.

2. $\widetilde{\tau}^*$ is the changing point for X_t.

3. $\mu_t^x = \overline{f}_t(t)$ for $0 \leq t \leq \widetilde{\tau}^*$.

Then

$$V(0,T) = \sup_{0\leq\tau\leq T} \mathbb{E}X(\tau) = \mathbb{E}X(\widetilde{\tau}^*).$$

Example 3.12

Given initial value G_0 and $f_0(u)$ for $0 \leq u \leq T$ and recall the definition $V_0(T)$.

According to above corollary, given any volatility surface $\beta_t(u)$, we can find a class of forward starting supermartingale X_t with $X_0 = G(0)$ which satisfies:

$$V(0,T) = \sup_{0\leq\tau\leq T} \mathbb{E}X(\tau).$$

Here we choose $\beta_t(u) = \sigma \cdot e^{-rt}$ and $\alpha_t(u) = 0$. Then we can get:

$$f_t(u) = f_0(u) + \sigma \cdot \int_0^t e^{-rs} dW_s$$

and

$$\tau^* = \inf\left\{0 \leq t \leq T | \int_t^T [f_0(u) + \sigma \cdot \int_0^t e^{-rs} dW_s] du \leq 0\right\},$$

which is equivalent to

$$\tau^* = \inf\left\{0 \leq t \leq T | \int_0^t e^{-rs} dW_s \leq -\frac{\int_t^T f_0(u)du}{\sigma(T-t)}\right\}.$$

Then we can construct $dX_t = \mu_t^x dt + \sigma_t^x dW_t$ wg satisfies:

$$\mu_t^x = \begin{cases} f_0(t) + \sigma \cdot \int_0^t e^{-rs} dW_s, & for \quad \mathrm{t} \leq \tau^* \\ \leq 0, & for \quad \mathrm{t} > \tau^* \end{cases}$$

In this case, the stopping time can be also written as:

$$\tau^* = \inf \left\{ 0 \leq t \leq T | \mu_t^x \leq f_0(t) - \frac{\int_t^T f_0(u) du}{T - t} \right\}$$

3.5 From Spot Drift To Forward Drift

In this subsection, given $dG_t = \mu_t dt + \sigma_t dW_t$ with $G(0)$. Let us consider the optimal stopping problem:

$$V(0) = \sup_{0 \leq \tau \leq T} \mathbb{E} G_\tau \tag{39}$$

For the bond market and European option market, given the spot rate model, one can first calculate $V(t, T)$ by taking expectation, and then use it to get the value of forward rate. For the American option market, it is difficult to compute the value of $V(t, T)$ given G_t by taking expectation over stopping times. An example where it is trivial to calculate is the American Call option, when the optimal stopping time is known to be the expiration time. Let us consider the stock selling problem in the following example.

Example 3.13

Stock process follows $dS_t = \rho S_t dt + \sigma S_t dW_t$ with initial value $S(0)$. Then the optimal stopping problem is

$$V(0) = \sup_{\tau \geq 0} \mathbb{E} G_\tau$$

with

$$G_t = e^{-rt}(S_t - a)$$

where a, r, ρ and σ are constants.

Then we can get:

$$dG_t = e^{-rt}[(\rho - r)S_t + ar]dt + e^{-rt}\sigma S_t dW_t.$$

For this infinite horizon problem, there usually exists constant boundary. In this case, the optimal time to sell stock is $\tau^* = \inf\{t \geq 0 | S(t) \geq b^*\}$. Then

$$f_t(t) = e^{-rt}[(\rho - r)S_t + ar]$$

and for $u > t$,

$$f_t(u) = \mathbb{E}_t \left\{ e^{-ru}[(\rho - r)S_u + ar]1(\max_{t \leq s \leq u} S_s < b^*) \right\}.$$

For the finite-time horizon stock selling problem, it is not easy to calculate the above expectation as the boundary is no longer a constant.

Normally there are more than one pair of volatility surface $\beta_t(u)$ and its admissible drift surface $\alpha_t^\beta(u)$ such that Spot Consistency Condition is satisfied:

$$\mu_t = f_t(t).$$

Definition 3.14 Recall Definition 3.4 for $\overline{f}_t(u)$. Define

$$\Sigma = \left\{ (\beta_t(u), \alpha_t^{\beta}(u)) : 0 \le t \le u \le T | \mu_t = \overline{f}_t(t) \right\}$$

and

$$F = \left\{ \overline{f}_t(u) | (\beta_t(u), \alpha_t^{\beta}(u)) \in \Sigma \right\}.$$

Definition 3.15 Recall Definition 3.5 for $\widetilde{\tau}^*$ associated to $\overline{f}_t(u)$. Define

$$\Gamma = \left\{ \widetilde{\tau}^* | \overline{f}_t(u) \in F \right\}.$$

Theorem 3.16

Suppose $|\mu_t| \le B$ for all $0 \le t \le T$ and for some constant B.

Then $\overline{\tau}^* = \operatorname{ess\,sup}_{\widetilde{\tau}^* \in \Gamma} \widetilde{\tau}^*$ is the largest optimal stopping time of the problem:

$$\sup_{0 \le \tau \le T} \mathbb{E} G_\tau = \mathbb{E} G_{\overline{\tau}^*} \tag{40}$$

Proof: Define

$$Y_t = \mathbb{E}_t G_{\overline{\tau}^*}. \tag{41}$$

Then Y_t is a martingale and $Y_{\overline{\tau}^*} = G_{\overline{\tau}^*}$.

There is a countable sequence $\tau_i \in \Gamma$ such that

$$\overline{\tau}^* = \operatorname{ess\,sup}_{i \ge 1} \tau_i.$$

Define: $Z_n = \tau_1 \vee \tau_2 \vee ... \vee \tau_n$, then it is easy to see $Z_n \nearrow \overline{\tau}^*$.

For a given $\overline{f}^i_t(u) \in F$ with associated τ_i, define

$$\widehat{V}^i(t,T) = G_t + \int_t^T \overline{f}^i_t(u)du. \tag{42}$$

Recall the property of $\widehat{V}^i(t,T)$, we have

$$\widehat{V}^i(\tau_i, T) = G_{\tau_i} \tag{43}$$

and by Theorem 3.10, for $0 \leq t \leq \tau_i$,

$$V(t,T) = \widehat{V}^i(t,T). \tag{44}$$

Therefore,

$$E_t G_{\tau_i} \geq G_t.$$

Define

$$Y_t^n = E_t G_{Z_n}.$$

Then

$$Y_t^1 = E_t G_{\tau_1} \geq G_t.$$

Suppose $Y_t^k \geq G_t$. Then

$$
\begin{aligned}
Y_t^{k+1} &= E_t[G_{Z_k} 1(Z_k = Z_{k+1}) + G_{\tau_{k+1}} 1(Z_k < Z_{k+1})] \\
&= E_t[Y_{Z_k} 1(\tau_{k+1} \leq Z_k) + G_{\tau_{k+1}} 1(\tau_{k+1} > Z_k)] \\
&= E_t[Y_{\tau_{k+1}} 1(\tau_{k+1} \leq Z_k) + G_{\tau_{k+1}} 1(\tau_{k+1} > Z_k)] \\
&= E_t G_{\tau_{k+1}} \\
&\geq G_t.
\end{aligned}
$$

We conclude $Y_t^n \geq G_t$ for all n. For $0 \leq t \leq Z_n$,

$$
\begin{aligned}
Y_t = \mathbb{E}_t G_{\overline{\tau}^*} &= \mathbb{E}_t G_{Z_n} + \mathbb{E}_t \int_{Z_n}^{\overline{\tau}^*} \mu_u du \\
&\geq \mathbb{E}_t G_{Z_n} - B\mathbb{E}_t[\overline{\tau}^* - Z_n] \\
&\geq G_t - B\mathbb{E}_t[\overline{\tau}^* - Z_n]
\end{aligned}
$$

Therefore we can get for $0 \leq t \leq \overline{\tau}^*$,

$$
Y_t \geq G_t. \tag{45}
$$

Since $\tau^* \leq \overline{\tau}^*$, we will have

$$
\sup_{0 \leq \tau \leq T} \mathbb{E} G_\tau = \mathbb{E} G_{\overline{\tau}^*}. \tag{46}
$$

$\diamond$

Example 3.17 Let us take another look of the example in the last subsection.

Suppose

$$\mu_t = f_0(t) + \sigma \cdot \int_0^t e^{-rs} dW_s$$

Then according to theorem above, we can get: the optimal stopping for this process satisfies:

$$\tau^* \geq \inf \left\{ 0 \leq t \leq T | \mu_t \leq f_0(t) - \frac{\int_t^T f_0(u)du}{T-t} \right\}.$$

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APPENDIX A: BOUNDARY VALUE PROBLEMS

Suppose $X = (X)_{0\leq t<\infty}$ is a strong Markov process with continuous paths in the probability space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P_x)$. Moreover, we assume X takes values in a measurable space $(R^d, \mathcal{B}(R^d))$ and $(\mathcal{F}_{t\geq 0})$ satisfies the usual condition.

For the boundary value problem, we refer Oksendal [23].Our goal in this subsection is to calculate $V(x) = E_x G(X_\tau)$ by using PDE method. We will first give the the PDE, which $V(x)$ should satisfy. Then we give the uniqueness theorems to prove the solution of the PDE $w(x)$ is also the solution of this expectation, i.e $w(x) = V(x)$. Now we assume D is a Borel set, τ_D is the first hitting time to D, i.e $\tau_D = inf\,\{t \geq 0 : X_t \in D\}$, $G : R^d \to R$ is a measurable function and $\lambda = (\lambda_t)_{t\geq 0}$ is given by $\lambda_t = \int_0^t \lambda(X_s)ds$ for a measurable continuous function $\lambda : R^d \to R$ and $L : R^d \to R$ is a continuous function. Then we have the following results:

1. Dirichlet Problem: If $V(x) = E_x G(X_{\tau_D})$ for $x \in R^d$, then the function V satisfies:

$$\mathcal{A}_X V = 0$$

   for all $x \in C$.

2. Killed Dirichlet Problem: If V(x)=E$_x e^{-\lambda_{\tau_D}} G(X_{\tau_D})$ for $x \in R^d$, then the function V satisfies:

$$\mathcal{A}_X V = \lambda V$$

   for all $x \in C$.

3. Poisson Problem: If $V(x) = E_x \int_0^{\tau_D} L(X_t)dt$ for $x \in R^d$, then the function V satis-

fies:

$$\mathcal{A}_X V = -L$$

for all $x \in C$.

4. Killed Poisson Problem: If $V(x) = E_x \int_0^{\tau_D} e^{-\lambda_t} L(X_t)dt$ for $x \in R^d$, then the function V satisfies:

$$\mathcal{A}_X V = \lambda V - L$$

for all $x \in C$.

Lemma 2.14 Define:

$$V(x) = E_x[\int_0^{\tau_D} e^{-rt} L(X_t)dt + e^{-r\tau_D} M(X_{\tau_D})]$$

where τ_D is the first hitting time to a Borel set D, X satisfies the setup. L is a continuous measurable function and M is measurable function. Then if characteristic operator exists for $V(x)$, then the following is true.

$$(\mathcal{A}_X - r)V(x) = -L(x)$$

for $x \in C$.

Proof: For any $x \in C$, which is the complement of set D, let U be an open set such that $x \in U \subset C$ and τ_{U^c} be the first hitting time to set U^c. Then it's easy to see that

$\tau_{U^c} \leq \tau_D$ P-a.s.

$$
\begin{aligned}
\mathbb{E}_x V(X_{\tau_{U^c}}) &= \mathbb{E}_x \mathbb{E}_{X_{\tau_{U^c}}} [\int_0^{\tau_D} e^{-rt} L(X_t)dt + e^{-r\tau_D} M(X_{\tau_D})] \\
&= \mathbb{E}_x \mathbb{E}_x \left\{ [\int_0^{\tau_D} e^{-rt} L(X_t)dt + e^{-r\tau_D} M(X_{\tau_D})] \circ \theta_{\tau_{U^c}} | \mathcal{F}_{\tau_{U^c}} \right\} \\
&= \mathbb{E}_x [\int_0^{\tau_D \circ \theta_{\tau_{U^c}}} e^{-rt} L(X_t \circ \theta_{\tau_{U^c}})dt + e^{-r\tau_D \circ \theta_{\tau_{U^c}}} M(X_{\tau_D} \circ \theta_{\tau_{U^c}})] \\
&= \mathbb{E}_x [\int_0^{\tau_D - \tau_{U^c}} e^{-rt} L(X_{t+\tau_{U^c}})dt + e^{-r(\tau_D - \tau_{U^c})} M(X_{\tau_D})] \\
&= \mathbb{E}_x [\int_{\tau_{U^c}}^{\tau_D} e^{-r(t-\tau_{U^c})} L(X_t)dt + e^{-r(\tau_D - \tau_{U^c})} M(X_{\tau_D})] \\
&= \mathbb{E}_x [\int_0^{\tau_D} e^{-r(t-\tau_{U^c})} L(X_t)dt - \int_0^{\tau_{U^c}} e^{-r(t-\tau_{U^c})} L(X_t)dt + e^{-r(\tau_D - \tau_{U^c})} M(X_{\tau_D})]
\end{aligned}
$$

According to the definition of characteristics operator, we can get:

$$
\begin{aligned}
\mathcal{A}_X V(x) &= \lim_{U^c \downarrow x} \frac{E_x V(X_{\tau_{U^c}}) - V(x)}{E_x \tau_{U^c}} \\
&= rV(x) - L(x)
\end{aligned}
\tag{47}
$$

Therefore, we can conclude that:

$$
(\mathcal{A}_X - r)V(x) = -L(x) \tag{48}
$$

for all $x \in C$.Let $V \in C^2$, then we can prove $\mathcal{A}_X V(x)$ exists for Ito diffusion process and

$$
\mathcal{A}_X V(x) = \sum_{i=1}^{d} b_i \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} (\sigma\sigma^T)_i j \frac{\partial^2 V}{\partial x_i \partial x_j} \tag{49}
$$

for d dimension Ito diffusion process.

In fact, the Dirichlet problem already gives the PDE which the value function should satisfy if there exists an optimal stopping time. In general, if X is d dimension Markov process, we will get a d dimension PDE. However, if $G(x)$ has some special form, we may have PDE with lower dimensions because of the property of the function. For example, if $Y_t = (t, X_t)$, the PDE should have form $L_Y = \frac{\partial}{\partial t} + L_X$. This is true for all $V(t,x)$ satisfying $V(0,x) = E_{(0,x)}G(\tau_D, X_{\tau_D})$. If we consider $G(t,x) = e^{-rt}G(x)$, then $L_Y = L_X - \lambda$.

We will give the unique theorem below. Let C be a open connected set in R^d, $M \in C(\partial C)$ and $L \in C(C)$. L_X is the generator of Ito diffusion process. i.e $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, where B_t is d dimensional Brownian motion. Moreover, we assume $\mu(x)$ and $\sigma(x)$ are continuous functions satisfying the existence of the SDE. Then the combined Dirichlet-Poisson problem is:

$$L_X w = -L \tag{50}$$

for $x \in C$ and

$$\lim_{x \to y, x \in C} w(x) = M(x) \tag{51}$$

for $y \in \partial C$. L_X is the generator of Ito diffusion process. i.e $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, where B_t is d dimensional Brownian motion. Moreover, we assume $\mu(x)$ and $\sigma(x)$ are continuous functions satisfying the existence of the SDE.

Theorem 2.15 (Uniqueness theorem)

Suppose the following statements are true:

1. M is bounded.

2. L satisfies $E_x[\int_0^{\tau_D} |L(X_t)|\, dt] < \infty$.

3. $\tau_D < \infty$ P^xa.s. for all x.

Then if $w \in C^2(C)$ is a bounded solution of the combined Dirichlet-Poisson problem above, we have

$$w(x) = E^x[M(X_{\tau_D})] + E^x[\int_0^{\tau_D} L(X_t)dt]$$

Proof can be found in [22].

Discounted and Integral Value Function

In this section, we assume the process $(X_t)_{0\leq t<\infty}$ satisfying:

$$dX_t = \mu t + dB_t$$

where, $(B_t)_{0\leq t<\infty}$ is 1-dimensional Brownian motion and μ is a constant.

Discounted and Integral Gain Function with One Stopping Time

The optimal stopping problem is:

$$V(x) = \sup_{0\leq\tau<\infty} E_x[\int_0^{\tau} e^{-rt}L(X_t)dt + e^{-r\tau}M(X_\tau)] \tag{52}$$

where X_t is 1-d drifted Brownian motion. In addition, L is continuous function and M is measurable function.

Let us first think of a special kind of stopping time. i.e. $\tau_a^b = inf\,\{t \geq 0 : X_t \in (-\infty, a] \cup [b, \infty)\}$

for some $a < b$. According to the theorem in the first chapter, we know $V(x) = E_x[\int_0^{\tau_a^b} e^{-rt} L(X_t)dt + e^{-r\tau_a^b} M(X_{\tau_a^b})]$ must satisfy the following PDE:

$$(\mu \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} - r)V(x) = -L(x) \tag{53}$$

for $x \in (a, b)$. We assume the following boundary conditions:

$$\begin{aligned} V(a) &= M(a) \\ V(b) &= M(b) \end{aligned} \tag{54}$$

The general solution of this PDE is:

$$V(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + q(x) \tag{55}$$

where r_i is the solution of the following equation:

$$\frac{1}{2} r_i^2 + \mu r_i - r = 0 \tag{56}$$

and $r_1 < 0 < r_2$. And

$$C_1 = \frac{(M(a) - q(a))e^{r_2 b} - (M(b) - q(b))e^{r_2 a}}{e^{r_2 b + r_1 a} - e^{r_1 b + r_2 a}} \tag{57}$$

$$C_2 = \frac{(M(a) - q(a))e^{r_1 b} - (M(b) - q(b))e^{r_1 a}}{e^{r_2 a + r_1 b} - e^{r_2 b + r_1 a}} \tag{58}$$

Lemma Suppose the following statements are true:

1. $M(x)$ and $q(x)$ are finite for any $x \in R$.

2. $\lim_{a\to-\infty}\frac{M(a)-q(a)}{e^{r_1 a}}=0.$

3. $\lim_{b\to\infty}\frac{M(b)-q(b)}{e^{r_2 b}}=0.$

Then if $a \to -\infty$

$$C_1 \to 0 \text{ and } C_2 \to [M(b)-q(b)]e^{-r_2 b}$$

If $b \to \infty$, then

$$C_1 \to [M(a)-q(a)]e^{-r_1 a} \text{ and } C_2 \to 0$$

Proposition Let $\tau_b = \inf\{t \geq 0 : X_t \in [b,\infty)\}$, consider the following problem:

$$V_b(x) = E_x[\int_0^{\tau_b} e^{-rt}L(X_t)dt + e^{-r\tau_b}M(X_{\tau b})]$$

where X_t is 1-d drifted Brownian motion with $\mu > 0$.L is continuous function and M is measurable function. In addition $M(x)$ and $q(x)$ is finite for all $x \in R$. Moreover $\lim_{a\to-\infty}\frac{M(a)-q(a)}{e^{r_1 a}}=0$. Then

$$V_b(x) = [M(b)-q(b)]e^{r_2(x-b)} + q(x)$$

for $x \in (-\infty, b)$.

Proposition Let $\tau_a = inf\{t \geq 0 : X_t \in (-\infty, a]\}$, consider the following problem:

$$V_a(x) = E_x[\int_0^{\tau_a} e^{-rt}L(X_t)dt + e^{-r\tau_a}M(X_{\tau a})]$$

where X_t is 1-d drifted Brownian motion with $\mu < 0$.L is continuous function and M is measurable function. In addition $M(x)$ and $q(x)$ is finite for all $x \in R$. Moreover,

$\lim_{a\to-\infty}\frac{M(a)-q(a)}{e^{r_1 a}}=0$. Then

$$V_a(x)=[M(a)-q(a)]e^{r_1(x-a)}+q(x)$$

for $x\in(a,\infty)$.

Now, let's define $\tau_a^t = inf\,\{s\ge t: X_s\in(-\infty,a]\}$ and

(59)
$$\begin{pmatrix} dZ_{1t} \\ dZ_{2t} \end{pmatrix} = \begin{pmatrix} e^{-rt}L(X_t) \\ \mu \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_t$$

Moreover, if we define $V_a(t,z_1,z_2)$ as the following:

(60)
$$\begin{aligned} V_a(t,z_1,z_2) &= E_{(t,z_1,z_2)}[\textstyle\int_0^{\tau_a^t} e^{-rt}L(X_t)dt + e^{-r\tau_a^t}M(X_{\tau a}^t)] \\ &= E_{(t,z_1,z_2)}[\textstyle\int_0^t e^{-rs}L(X_s)ds + \int_t^{\tau_a^t} e^{-rs}L(X_s)ds + e^{-rt}e^{-r(\tau_a^t-t)}M(X_{\tau a}^t)] \\ &= z_1 + e^{-rt}[\textstyle\int_0^{\tau_a} e^{-rs}L(X_s)ds + e^{-r(\tau_a)}M(X_{\tau a})] \\ &= z_1 + e^{-rt}V_a(z_2) \end{aligned}$$

According to the sufficient theorem in the first chapter and lemma above, we can get the following theorems.

Proposition 4.1 Suppose $\mu \geq 0$, if $M(x) \in C^2(R)$ has the following properties for some b:

1. $M^{'}(b_+) = [M(b) - q(b)]r_2 + q^{'}(b)$. (Smooth Pasting)
2. $(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})M(x) + L(x) \leq 0$ for $x \geq b$. (Superharmonic Property)
3. $[M(b) - q(b)]e^{-r_2 b} \geq [M(x) - q(x)]e^{-r_2 x}$, for all $x \in (-\infty, b)$.(Dominating Property)

then

$$\widehat{V}(t, z_1, z_2) = \begin{cases} z_1(t) + e^{-rt}[M(b) - q(b)]e^{r_2(z_2(t)-b)} + e^{-rt}q(z_2(t)), & for \quad z_2(t) \in (-\infty, b) \\ z_1(t) + e^{-rt}M(z_2(t)), & for \quad z_2(t) \in [b, \infty) \end{cases}$$

and $\tau = inf\,\{t \geq 0 : X_t \in [b, \infty)\}$ is an optimal stopping time.

Proposition 4.2 Suppose $\mu \leq 0$, if $M(x) \in C^2(R)$ has the following properties for some b:

1. $M^{'}(a_-) = [M(a) - q(a)]r_1 + q^{'}(a)$. (Smooth Pasting)
2. $(-r + \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})M(x) + L(x) \leq 0$ for $x \leq a$. (Superharmonic Property)
3. $[M(a) - q(a)]e^{-r_1 a} \geq [M(x) - q(x)]e^{-r_1 x}$, for all $x \in (a, \infty)$.(Dominating Property)

then

$$\widehat{V}(t, x) = \begin{cases} e^{-rt}M(x), & for \quad x \in (-\infty, a] \\ e^{-rt}[M(a) - q(a)]e^{r_1(x-a)} + e^{-rt}q(x), & for \quad x \in (a, \infty) \end{cases}$$

and $\tau = \inf\{t \geq 0 : X_t \in (-\infty, a]\}$ is an optimal stopping time.